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Supersymmetry in noncommutative superspaces

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Abstract: Non commutative superspaces can be introduced as the Moyal-Weyl quantization of a Poisson bracket for classical superfields. Different deformations are studied corresponding to constant background fields in string theory. Supersymmetric and non supersymmetric deformations can be defined, depending on the differential operators used to define the Poisson bracket. Some examples of deformed, 4 dimensional lagrangians are given. For extended superspace ($N > 1$), some new deformations can be defined, with no analogue in the $N = 1$ case.

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1. Introduction

Non commutative geometry and supergeometry naturally arise in string theory in several contexts. Among others, we may mention the work of Connes, Douglas and Schwarz [1] where non commutative tori where introduced as possible compactification spaces of M theory, the work of Banks, Fischler, Shenker and Susskind [2] where M-theory was related to the $N \to \infty$ limit of the supersymmetric matrix quantum mechanics describing D0-branes and the work of Seiberg and Witten [3] where a certain limit of the string dynamics is described by a gauge theory in presence of a non zero background field $B_{\mu\nu}$. (For a review, see ref. [4], and references therein).

More recently, the extension of non commutativity to odd variables has been related to the presence of other background fields. The R-R field strength backgrounds give rise to a deformation of type $\theta - \theta$ [5], and the gravitino background gives rise to an $x - \theta$ deformation [6].

Field theories in non commutative spaces have been considered in the literature in a broader context, where a huge amount of work has been done. For an introduction to different aspects of the subject, see for example refs. [7]–[12].

In connection with string theory, supersymmetric theories have been considered mainly where the deformation of superspace affects only the spacetime part ($x - x$ deformations). There, the extension of Wess-Zumino [13, 14] and Yang-Mills models is straightforward once the gauge invariance in superspace is properly defined [14]. Also in ref. [14] the possibility of a supersymmetric deformation of superspace with non vanishing $\theta - \theta$ deformation was explored. In fact, the possibility of having fermi coordinates that have a non zero anticommutator was already studied in the literature [15, 16]. From a more mathematical point of view, non commutative supermanifolds and supervarieties have also been considered in the literature [17]–[19].

Starting from the observation of Ooguri and Vafa [5], deformations of the anticommuting variables have acquired renewed interest. The effect of such deformations in the lagrangian has been investigated by Seiberg [20] and Berkovits and Seiberg [21].

In the present paper we consider a variety of deformations, both for $N = 1$ and extended ($N = 2$) supersymmetry in $D = 4$. These deformations vary according the differential operators chosen to construct the Poisson bracket that afterwards becomes quantized with a star product of Moyal-Weyl type. In particular, we show explicitly the difference between the deformation considered in ref. [14] and the one proposed in ref. [20]. The first one has the advantage of being manifestly supersymmetric, the second one, although it explicitly breaks one half of the supersymmetry, allows the definition of chiral and antichiral superfields, which form subalgebras of the star product. This was not possible with the supersymmetric deformation, and allows a simple generalization of super Yang-Mills theories to the deformed superspace. For the Wess-Zumino model both deformations lead to the same lagrangian, preserving 1/2 of the supersymmetry.

We also consider deformations of type $x - \theta$ and explore the consequences of these in the lagrangian for some simple cases.

Finally we study deformations of extended superspaces where new possibilities arise. For example, one can consistently have a non trivial $\theta - \bar{\theta}$ anticommutator, which is related to having constant vector backgrounds. This corresponds to deformations of harmonic superspace [22].

The paper is organized as follows. In section 2 we remind some known facts about Poisson brackets in superspace and fix the notation. Section 3 is devoted to the definition of supersymmetric Poisson brackets and their Moyal-Weyl deformation. In particular, we obtain the formula for the Moyal-Weyl deformation in presence of an $x - \theta$ deformation. In section 4 we consider extended superspace and its deformations. In section 5 we study the deformed Wess-Zumino model in different scenarios.

2. Generalities on super Poisson brackets

We consider the superalgebra $\mathcal{A} = C^\infty(\mathbb{R}^n) \otimes \Lambda(\mathbb{R}^m)$ which has an obvious $\mathbb{Z}_2$ grading. We say that an even element ϕ has parity $p(\phi) = p_\phi = 0$ and an odd element has parity $p(\phi) = p_\phi = 1$. We have that

$$p(\phi\psi) = p(\phi) + p(\psi) \bmod (2)$$

Even and odd elements are homogeneous elements.

A Poisson bracket on $\mathcal{A}$ is a bilinear operation

$$\{\ ,\ \} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

such that for homogeneous elements ϕ, ψ and χ

1. $p(\{\phi, \psi\}) = p(\phi) + p(\psi)$ (it is an *even* Poisson bracket).
2. $\{\phi, \psi\} = -(-1)^{p_\phi p_\psi}\{\psi, \phi\}$.
3. (Derivation property)

$$\{\phi\phi', \psi\} = \phi\{\phi', \psi\} + (-1)^{p'_\phi p_\psi}\{\phi, \psi\}\phi'$$
$$\{\phi, \psi\psi'\} = (-1)^{p_\psi p_\phi}\phi\{\phi, \psi'\} + \{\phi, \psi\}\psi' \,.$$

4. (Graded Jacobi identity)

$$\{\phi, \{\psi, \chi\}\} + (-1)^{p_\chi(p_\phi + p_\psi)}\{\chi, \{\phi, \psi\}\} + (-1)^{p_\phi(p_\psi + p_\chi)}\{\psi, \{\chi, \phi\}\} = 0 \,.$$

Let us denote

$$A = a \,, \qquad \text{for} \quad A = 1, \dots n \,,$$
$$A = \alpha + n \,, \qquad \text{for} \quad A = n+1, \dots n+m.$$

z^A denotes generically all variables x^a and θ^α. Then, a Poisson bracket on $\mathcal{A}$ can be written as

$$\{\phi, \psi\} = \phi \overleftarrow{\partial}_A P^{AB} \overrightarrow{\partial}_B \psi \,, \tag{2.1}$$

where

$$\overleftarrow{\partial}_A = \frac{\overleftarrow{\partial}}{\partial z^A} \,, \qquad \overrightarrow{\partial}_A = \frac{\overrightarrow{\partial}}{\partial z^A}$$

are respectively right and left derivatives. They both coincide with the ordinary derivative in the case that z^A is an even variable. For odd variables one has

$$(f(x)\theta^{\alpha_1} \dots \theta^{\alpha_k}\xi\theta^{\beta_1} \dots \theta^{\beta_l})\overleftarrow{\partial}_\xi = (-1)^l (f(x)\theta^{\alpha_1} \dots \theta^{\alpha_k}\theta^{\beta_1} \dots \theta^{\beta_l}) \,,$$
$$\overrightarrow{\partial}_\xi (f(x)\theta^{\alpha_1} \dots \theta^{\alpha_k}\xi\theta^{\beta_1} \dots \theta^{\beta_l}) = (-1)^k (f(x)\theta^{\alpha_1} \dots \theta^{\alpha_k}\theta^{\beta_1} \dots \theta^{\beta_l}) \,,$$

so

$$\phi \overleftarrow{\partial}_A = (-1)^{p_A(p_\phi + 1)} \overrightarrow{\partial}_A \phi \,.$$

They are odd derivations of the algebra $\mathcal{A}$:

$$\overrightarrow{\partial}_\xi(\phi\psi) = \overrightarrow{\partial}_\xi(\phi)\psi + (-1)^{p_\xi p_\phi}\phi \overrightarrow{\partial}_\xi(\psi), \qquad \text{(left derivation)} \,,$$
$$(\phi\psi)\overleftarrow{\partial}_\xi = \phi(\psi)\overleftarrow{\partial}_\xi + (-1)^{p_\xi p_\psi}(\phi)\overleftarrow{\partial}_\xi \psi, \qquad \text{(right derivation)}.$$

Right and left derivatives are chosen in such way that for $F \in \mathcal{A}$

$$dF = F\overleftarrow{\partial}_\xi d\theta = d\theta \overrightarrow{\partial}_\xi F \,.$$

If $P^{AB} = -(-1)^{p_A p_B} P^{BA}$ and $p(P^{AB}) = p_A + p_B$, then properties 1,2,3 are automatically satisfied.

If we want to consider a constant Poisson bracket (P^{AB} = constant, that is, independent of z^A), we have to extend the scalars to a commutative superalgebra. The Poisson tensor can be written in coordinates as a matrix with blocks

$$\begin{pmatrix} T_{ab} & R_{a\beta} \\ S_{\alpha b} & U_{\alpha\beta}. \end{pmatrix}$$

Then the entries of R and S are *odd scalars.* The algebra will be $\mathcal{A}[\xi^1, \dots \xi^s] = \mathbb{C}^\infty(\mathbb{R}^n) \otimes \Lambda(\mathbb{R}^m) \otimes \Lambda(\mathbb{R}^s)$, where the odd generators $\xi^i \in \Lambda(\mathbb{R}^s)$ are considered as scalars (inert under the derivations). Then, the graded Jacobi identity is also satisfied.

3. Supersymmetric Poisson brackets and star products

We consider a four dimensional space time with coordinates x^μ, $\mu = 0, \dots 3$ and minkowskian signature. For the moment being we take $N = 1$ supersymmetry, with one complex Weyl spinor θ^α, $\alpha = 1, 2$, generating the odd part of the superspace. On the space of superfields the covariant left derivatives are defined as

$$\begin{aligned} \overrightarrow{D}_\alpha \Phi &= \overrightarrow{\partial}_\alpha \Phi + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\Phi \\ \overrightarrow{\bar D}_{\dot\alpha} \Phi &= -\overrightarrow{\bar\partial}_{\dot\alpha} \Phi - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\Phi \end{aligned}$$

satisfying an algebra

$$\begin{aligned} \{\overrightarrow{D}_\alpha, \overrightarrow{D}_\beta\} &= \{\overrightarrow{\bar D}_{\dot\alpha}, \overrightarrow{\bar D}_{\dot\beta}\} = 0 \\ \{\overrightarrow{D}_\alpha, \overrightarrow{\bar D}_{\dot\alpha}\} &= -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\,. \end{aligned}$$

Given a left derivation $\overrightarrow{\mathcal{D}}$ of degree $p_\mathcal{D}$, that is,

$$\overrightarrow{\mathcal{D}}(\Phi\Psi) = \overrightarrow{\mathcal{D}}(\Phi)\Psi + (-1)^{p_\mathcal{D}p_\Phi}\Phi\overrightarrow{\mathcal{D}}(\Psi)\,,$$

one can define a right derivation $\overleftarrow{\mathcal{D}}$, also of degree $p_\mathcal{D}$

$$(\Phi\Psi)\overleftarrow{\mathcal{D}} = \Phi(\Psi)\overleftarrow{\mathcal{D}} + (-1)^{p_\mathcal{D}p_\Psi}(\Phi)\overleftarrow{\mathcal{D}}\Psi\,,$$

in the following way

$$\Phi\overleftarrow{\mathcal{D}} = (-1)^{p_\mathcal{D}(p_\Phi+1)}\overrightarrow{\mathcal{D}}\Phi\,.$$

Then we can define the right covariant derivatives as

$$\begin{aligned} \Phi\overleftarrow{D}_\alpha &= \Phi\overleftarrow{\partial}_\alpha - i\partial_\mu\Phi\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} \\ \Phi\overleftarrow{\bar D}_{\dot\alpha} &= -\Phi\overleftarrow{\bar\partial}_{\dot\alpha} + i\partial_\mu\Phi\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\,. \end{aligned}$$

They satisfy an algebra

$$\begin{aligned} \{\overleftarrow{D}_\alpha, \overleftarrow{D}_\beta\} &= \{\overleftarrow{\bar D}_{\dot\alpha}, \overleftarrow{\bar D}_{\dot\beta}\} = 0 \\ \{\overleftarrow{D}_\alpha, \overleftarrow{\bar D}_{\dot\alpha}\} &= +2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\,. \end{aligned}$$

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On may notice that for any two odd derivations

$$[\overleftarrow{D},\overrightarrow{D'}]\Phi = (-1)^{p_\Phi}\{\overrightarrow{D},\overrightarrow{D'}\}\Phi\,. \tag{3.1}$$

The supersymmetry algebra is realized as an algebra of left derivations on superspace

$$\overrightarrow{Q}_\alpha = \overrightarrow{\partial}_\alpha - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\,, \qquad \overrightarrow{\bar Q}_\alpha = -\overrightarrow{\bar\partial}_{\dot\alpha} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\,,$$

satisfying

$$\begin{aligned}\{\overrightarrow{Q}_\alpha,\overrightarrow{Q}_\beta\} &= \{\overrightarrow{\bar Q}_{\dot\alpha},\overrightarrow{\bar Q}_{\dot\beta}\} = 0\\ \{\overrightarrow{Q}_\alpha,\overrightarrow{\bar Q}_{\dot\alpha}\} &= +2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\,.\end{aligned}$$

One has also

$$\{\overrightarrow{D}_\alpha,\overrightarrow{Q}_\beta\} = \{\overrightarrow{D}_\alpha,\overrightarrow{\bar Q}_{\dot\beta}\} = \{\overrightarrow{\bar D}_{\dot\alpha},\overrightarrow{Q}_\beta\} = \{\overrightarrow{\bar D}_{\dot\alpha},\overrightarrow{\bar Q}_{\dot\beta}\} = 0\,. \tag{3.2}$$

3.1 Supersymmetric Poisson brackets

On the space of complex superfields one can consider the following Poisson bracket, which is a generalization of the Poisson bracket considered in [14]:

$$\{\Phi,\Psi\}_1 = P^{\mu\nu}\frac{\partial\Phi}{\partial x^\mu}\frac{\partial\Psi}{\partial x^\nu} + P^{\alpha\beta}\Phi\overleftarrow{D}_\alpha\overrightarrow{D}_\beta\Psi + \frac{\partial\Phi}{\partial x^\mu}P^{\mu\alpha}\overrightarrow{D}_\alpha\Psi - \Phi\overleftarrow{D}_\alpha P^{\mu\alpha}\frac{\partial\Psi}{\partial x^\mu}\,. \tag{3.3}$$

This satisfies the Jacobi identity with $P^{\mu\nu}$ antisymmetric, $P^{\alpha\beta}$ symmetric, $P^{\mu\alpha}$ arbitrary (and odd), and all of them constant. It will be convenient to take $P^{\mu\nu}$ pure imaginary, while the other matrices are just complex. Because of (3.1) and (3.2), $\{\ ,\ \}_1$ is a *supersymmetric* Poisson bracket, that is, the supersymmetry charges are derivations with respect to the Poisson bracket,

$$\begin{aligned}\overrightarrow{Q}\{\Phi,\Psi\} &= \left\{\overrightarrow{Q}\Phi,\Psi\right\} + (-1)^{p_\Phi}\left\{\Phi,\overrightarrow{Q}\Psi\right\}\\ \overrightarrow{\bar Q}\{\Phi,\Psi\} &= \left\{\overrightarrow{\bar Q}\Phi,\Psi\right\} + (-1)^{p_\Phi}\left\{\Phi,\overrightarrow{\bar Q}\Psi\right\}.\end{aligned}$$

One can replace D by $\bar D$ in (3.3) and write another Poisson bracket

$$\{\Phi,\Psi\}_2 = P^{\mu\nu}\frac{\partial\Phi}{\partial x^\mu}\frac{\partial\Psi}{\partial x^\nu} + P^{\dot\alpha\dot\beta}\Phi\overleftarrow{\bar D}_{\dot\alpha}\overrightarrow{\bar D}_{\dot\beta}\Psi + \frac{\partial\Phi}{\partial x^\mu}P^{\mu\dot\alpha}\overrightarrow{\bar D}_{\dot\alpha}\Psi - \Phi\overleftarrow{\bar D}_{\dot\alpha}P^{\mu\dot\alpha}\frac{\partial\Psi}{\partial x^\mu}\,,$$

but one cannot have in principle terms with both derivatives, D and $\bar D$, since they do not anticommute, and the Jacobi identity will not be immediately satisfied. A consequence of this is that $\{\ ,\ \}_{1,2}$ are degenerate in the space of odd variables. Another consequence is that in minkowskian space the Poisson brackets do not satisfy a reality condition,

$$\{\Phi^*,\Psi^*\}_{1,2} \neq \{\Psi,\Phi\}^*_{1,2}\,,$$

which does not mean that they are ill defined nor inconsistent in any way. If we take $(P^{\alpha\beta})^* = P^{\dot\alpha\dot\beta}$ and $(P^{\mu\alpha})^* = -P^{\mu\dot\alpha}$, they satisfy instead the relation

$$\{\Phi^*,\Psi^*\}_{1,2} = \{\Psi,\Phi\}^*_{2,1}\,.$$

In section 5.4 we will use the quantization of these Poisson brackets to give a prescription for a deformed Wess-Zumino lagrangian which is invariant with respect to the whole supertranslation algebra.

In other signatures or dimensions, a similar, supersymmetric Poisson bracket may admit such a reality condition [24].

Chiral superfields, $\bar{D}_{\dot{\alpha}}\Phi = 0$, are a Poisson subalgebra of $\{\ ,\ \}_2$ (as antichiral superfields, $D_{\dot{\alpha}}\Phi = 0$, are a Poisson subalgebra of $\{\ ,\ \}_1$). Indeed, the Poisson structures restricted to these subspaces involve only the even coordinates x^μ.

It is useful to express $\{\ ,\ \}_{1,2}$ in terms of ordinary derivatives:

$$\begin{aligned}
\{\Phi,\Psi\}_1 &= \left(P^{\mu\nu} + C^{\mu\nu}\bar{\theta}\bar{\theta} + iP^{\mu\alpha}\sigma^\nu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} - iP^{\nu\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\right)\partial_\mu\Phi\partial_\nu\Psi + \\
&\quad + \partial_\mu\Phi\left(iP^{\alpha\beta}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + P^{\mu\beta}\right)\overrightarrow{\partial}_\beta\Psi - \Phi\overleftarrow{\partial}_\alpha\left(iP^{\alpha\beta}\sigma^\nu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}} + P^{\nu\alpha}\right)\partial_\nu\Psi + \\
&\quad + P^{\alpha\beta}\Phi\overleftarrow{\partial}_\alpha\overrightarrow{\partial}_\beta\Psi\,, \\
\{\Phi,\Psi\}_2 &= \left(P^{\mu\nu} + \theta\theta D^{\mu\nu} - iP^{\mu\dot{\alpha}}\theta^\alpha\sigma^\nu_{\alpha\dot{\alpha}} + iP^{\mu\dot{\alpha}}\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\right)\partial_\mu\Phi\partial_\nu\Psi + \\
&\quad + \partial_\mu\Phi\left(iP^{\dot{\alpha}\dot{\beta}}\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}} - P^{\mu\dot{\beta}}\right)\overrightarrow{\bar{\partial}}_{\dot{\beta}}\Psi - \Phi\overleftarrow{\bar{\partial}}_{\dot{\alpha}}\left(iP^{\dot{\alpha}\dot{\beta}}\theta^\beta\sigma^\nu_{\beta\dot{\beta}} - P^{\nu\dot{\alpha}}\right)\partial_\nu\Psi + \\
&\quad + P^{\dot{\alpha}\dot{\beta}}\Phi\overleftarrow{\bar{\partial}}_{\dot{\alpha}}\overrightarrow{\bar{\partial}}_{\dot{\beta}}\Psi\,,
\end{aligned}$$

where

$$C^{\mu\nu} = \frac{1}{2}P^{\alpha\beta}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}}\,, \qquad D^{\mu\nu} = -\frac{1}{2}P^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}\,.$$

From these formulas the commutation rules of the basic variables x^μ, θ^α, $\bar{\theta}^{\dot{\alpha}}$ can be read directly.

We make now a change of variables [25]

$$x^\mu\,,\ \theta^\alpha\,,\ \bar{\theta}^{\dot{\alpha}} \ \longrightarrow\ y^\mu = x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\,,\ \theta^\alpha,\ \bar{\theta}^{\dot{\alpha}}\,.$$

A superfield may be expressed in both coordinate systems

$$\Phi(x,\theta,\bar{\theta}) = \Phi'(y,\theta,\bar{\theta})\,.$$

The covariant derivatives and supersymmetry charges take the form

$$\begin{aligned}
\overrightarrow{D}_\alpha\Phi' &= \overrightarrow{\partial}_\alpha\Phi' + 2i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial\Phi'}{\partial y^\mu} & \overrightarrow{\bar{D}}_{\dot{\alpha}}\Phi' &= -\overrightarrow{\bar{\partial}}_{\dot{\alpha}}\Phi'\,, \\
\overrightarrow{\bar{Q}}_{\dot{\alpha}}\Phi' &= -\overrightarrow{\bar{\partial}}_{\dot{\alpha}}\Phi' + 2i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial\Phi'}{\partial y^\mu} & \overrightarrow{Q}_\alpha\Phi' &= \overrightarrow{\partial}_\alpha\Phi'\,.
\end{aligned}$$

In the new coordinates the brackets $\{\ ,\ \}_{1,2}$ become

$$\begin{aligned}
\{\Phi',\Psi'\}_1 &= \left(P^{\mu\nu} + 4C^{\mu\nu}\bar{\theta}\bar{\theta} + 2iP^{\mu\alpha}\sigma^\nu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} - 2iP^{\nu\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\right)\frac{\partial\Phi'}{\partial y^\mu}\frac{\partial\Psi'}{\partial y^\nu} + \\
&\quad + \frac{\partial\Phi'}{\partial y^\mu}\left(2iP^{\alpha\beta}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + P^{\mu\beta}\right)\overrightarrow{\partial}_\beta\Psi' - \Phi'\overleftarrow{\partial}_\alpha\left(2iP^{\alpha\beta}\sigma^\nu_{\beta\dot{\beta}}\bar{\theta}^{\dot{\beta}} + P^{\nu\alpha}\right)\frac{\partial\Psi'}{\partial y^\nu} + \\
&\quad + P^{\alpha\beta}\Phi'\overleftarrow{\partial}_\alpha\overrightarrow{\partial}_\beta\Psi', \\
\{\Phi',\Psi'\}_2 &= P^{\mu\nu}\frac{\partial\Phi'}{\partial y^\mu}\frac{\partial\Psi'}{\partial y^\nu} + P^{\dot{\alpha}\dot{\beta}}\Phi'\overleftarrow{\bar{\partial}}_{\dot{\alpha}}\overrightarrow{\bar{\partial}}_{\dot{\beta}}\Psi' + \frac{\partial\Phi'}{\partial y^\mu}P^{\mu\dot{\alpha}}\overrightarrow{\bar{\partial}}_{\dot{\alpha}}\Psi' - \Phi'\overleftarrow{\bar{\partial}}_{\dot{\alpha}}P^{\mu\dot{\alpha}}\frac{\partial\Psi'}{\partial y^\mu}\,,
\end{aligned}$$

which simplifies $\{\ ,\ \}_2$.

3.2 Non supersymmetric Poisson brackets

One can define different Poisson brackets by making use of the operators Q_α and $\bar{Q}_{\dot\alpha}$. In fact, the operators D's and the Q's play interchangeable roles. Consider for example the brackets [20]

$$\begin{aligned}
\{\Phi,\Psi\}_3 &= P^{\mu\nu}\frac{\partial\Phi}{\partial x^\mu}\frac{\partial\Psi}{\partial x^\nu} + P^{\alpha\beta}\Phi\overleftarrow{Q}_\alpha\overrightarrow{Q}_\beta\Psi + \frac{\partial\Phi}{\partial x^\mu}P^{\mu\alpha}\overrightarrow{Q}_\alpha\Psi - \Phi\overleftarrow{Q}_\alpha P^{\mu\alpha}\frac{\partial\Psi}{\partial x^\mu} \\
\{\Phi,\Psi\}_4 &= P^{\mu\nu}\frac{\partial\Phi}{\partial x^\mu}\frac{\partial\Psi}{\partial x^\nu} + P^{\dot\alpha\dot\beta}\Phi\overleftarrow{\bar{Q}}_{\dot\alpha}\overrightarrow{\bar{Q}}_{\dot\beta}\Psi + \frac{\partial\Phi}{\partial x^\mu}P^{\mu\dot\alpha}\overrightarrow{\bar{Q}}_{\dot\alpha}\Psi - \Phi\overleftarrow{\bar{Q}}_{\dot\alpha}P^{\mu\dot\alpha}\frac{\partial\Psi}{\partial x^\mu}\,,
\end{aligned} \tag{3.4}$$

where the right acting charges are

$$\begin{aligned}
\Phi\overleftarrow{Q}_\alpha &= \Phi\overleftarrow{\partial}_\alpha + i\frac{\partial\Phi}{\partial x^\mu}\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} \\
\Phi\overleftarrow{\bar{Q}}_{\dot\alpha} &= \Phi\overleftarrow{\bar\partial}_{\dot\alpha} + i\frac{\partial\Psi}{\partial x^\mu}\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\,.
\end{aligned}$$

In terms of the coordinates $y,\theta,\bar\theta$, the bracket $\{\ ,\ \}_3$, for example, becomes

$$\{\Phi',\Psi'\}_3 = P^{\mu\nu}\frac{\partial\Phi'}{\partial y^\mu}\frac{\partial\Psi'}{\partial y^\nu} + P^{\alpha\beta}\Phi'\overleftarrow{\partial}_\alpha\overrightarrow{\partial}_\beta\Psi' + \frac{\partial\Phi}{\partial y^\mu}P^{\mu\alpha}\overrightarrow{\partial}_\alpha\Psi - \Phi\overleftarrow{\partial}_\alpha P^{\mu\alpha}\frac{\partial\Psi}{\partial y^\mu}\,.$$

The quantization of this bracket with $P^{\mu\nu} = P^{\mu\alpha} = 0$ was explored in ref. [20]. Since $\{Q,\bar{Q}\} \neq 0$, this bracket is not supersymmetric with respect to the charges $\bar{Q}$, that is,

$$\overrightarrow{\bar{Q}}\{\Phi,\Psi\}_3 \neq \left\{\overrightarrow{\bar{Q}}\Phi,\Psi\right\}_3 + (-1)^{p_\Phi}\left\{\Phi,\overrightarrow{\bar{Q}}\Psi\right\}_3\,,$$

although it is still supersymmetric with respect to the charges Q. Because of (3.2), one has instead that the operators D and $\bar{D}$ are derivations with respect to this bracket,

$$\begin{aligned}
\overrightarrow{D}\{\Phi,\Psi\}_{3,4} &= \left\{\overrightarrow{D}\Phi,\Psi\right\}_{3,4} + (-1)^{p_\Phi}\left\{\Phi,\overrightarrow{D}\Psi\right\}_{3,4} \\
\overrightarrow{\bar{D}}\{\Phi,\Psi\}_{3,4} &= \left\{\overrightarrow{\bar{D}}\Phi,\Psi\right\}_{3,4} + (-1)^{p_\Phi}\left\{\Phi,\overrightarrow{\bar{D}}\Psi\right\}_{3,4}\,.
\end{aligned}$$

It follows that the subspaces of chiral ($\bar{D}\Phi$=0) and antichiral ($D\Phi$=0) superfields are Poisson subalgebras of $\{\ ,\ \}_{3,4}$.

One could use a different change of variables

$$x^\mu,\ \theta^\alpha,\ \bar\theta^{\dot\alpha} \ \longrightarrow\ \bar{y}^\mu = x^\mu - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha},\ \theta^\alpha,\ \bar\theta^{\dot\alpha}\,,$$

with superfields

$$\Phi(x,\theta,\bar\theta) = \Phi''(\bar{y},\theta,\bar\theta)\,.$$

The brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_4$ would acquire simpler forms. The procedure is identical and we will not repeat it here.

3.3 Moyal-Weyl star products

Generically we consider Poisson brackets of the form

$$\{\Phi, \Psi\} = \Phi \overleftarrow{\mathcal{D}_A} P^{AB} \overrightarrow{\mathcal{D}_B} \Psi \,, \tag{3.5}$$

where the index A runs over all the variables, even and odd. The derivations $\overrightarrow{\mathcal{D}_A}$ commute (or anticommute) with each other and $\overrightarrow{\mathcal{D}_C} P^{AB} = 0$. The matrix P^{AB} has the right symmetry properties for (3.5) to be a Poisson bracket. Under these assumptions there is an associative star product of Moyal-Weyl type, defined by

$$\Phi \star \Psi = e^{hP}(\Phi, \Psi) = \sum_{n=0}^{\infty} \frac{h^n}{n!} P^n(\Phi, \Psi) \,, \tag{3.6}$$

where

$$P^n(\Phi, \Psi) = \sum_{A_1, \cdots A_n; B_1, \cdots B_n} (-1)^{\rho^{B_1 \cdots B_n}_{A_1 \cdots A_n}} \cdot \Phi \overleftarrow{\mathcal{D}}_{A_1} \overleftarrow{\mathcal{D}}_{A_2} \cdots \overleftarrow{\mathcal{D}}_{A_n} P^{A_1 B_1} P^{A_2 B_2} \cdots P^{A_n B_n} \overrightarrow{\mathcal{D}}_{B_n} \cdots \overrightarrow{\mathcal{D}}_{B_2} \overrightarrow{\mathcal{D}}_{B_1} \Psi \,,$$

and

$$\rho^{B_1 \cdots B_n}_{A_1 \cdots A_n} = \sum_{i=1}^{n-1} (p_{A_i} + p_{B_i}) \sum_{j=i+1}^{n} p_{A_j} \,.$$

The associativity of the star product

$$\phi \star (\psi \star \chi) = (\phi \star \psi) \star \chi \,,$$

follows from the associativity in the purely even case (see for example [23]).

The sign ρ it is needed to take into account the odd character of $P^{\mu\alpha}$. The procedure is as follows. We decompose again $A = (\mu, \alpha)$ and we define

$$\overrightarrow{\mathcal{K}}_a = \delta_{\mu a} P^{\mu\alpha} \overrightarrow{\mathcal{D}}_\alpha \,,$$

where a and μ run over the same set of numbers. Then $\overrightarrow{\mathcal{K}}_a$ is an even derivation. The Poisson bracket can be written as

$$\{\Phi, \Psi\} = P^{\mu\nu} \Phi \overleftarrow{\mathcal{D}}_\mu \overrightarrow{\mathcal{D}}_\nu \Psi + P^{\alpha\beta} \Phi \overleftarrow{\mathcal{D}}_\alpha \overrightarrow{\mathcal{D}}_\beta \Psi + \delta^{\mu a} \big(\Phi \overleftarrow{\mathcal{D}}_\mu \overrightarrow{\mathcal{K}}_a \Psi - \Phi \overleftarrow{\mathcal{K}}_a \overrightarrow{\mathcal{D}}_\mu \Psi \big) \,.$$

This is a Poisson bracket with $\tilde{P}^{AB}$ of block diagonal type (the index A now runs over (μ, a, α)). Then we can apply the standard formula for the Moyal-Weyl product (see for example ref. [14]) and obtain the quantization. Returning to the previous notation gives the sign ρ. In the case where P^{AB} is block diagonal (that is, P^{AB} is always even), then $\rho = 0$.

Notice that because of

$$\frac{\partial \Phi}{\partial x^\mu} = \frac{\partial \Phi'}{\partial y^\mu} = \frac{\partial \Phi''}{\partial \bar{y}^\mu} \,,$$

the star products defined by (3.6) in each of the three sets of coordinates $(x, \theta, \bar{\theta})$, $(y, \theta, \bar{\theta})$, $(\bar{y}, \theta, \bar{\theta})$ is exactly the same.

From the explicit formula for the quantization of the Poisson bracket (3.5), one has that if a derivation $\mathcal{K}$ (anti)commutes with $\mathcal{D}$, it is also a derivation either, of the Poisson bracket and of the corresponding star product,

$$\mathcal{K}(\Phi \star \Psi) = (\mathcal{K}\Phi) \star \Psi + (-1)^{p_{\mathcal{K}} p_\Phi} \Phi \star \mathcal{K}(\Psi)\,.$$

So a supersymmetric Poisson bracket gives rise to a supersymmetric star product.

We remark again that in minkowskian signature the star products defined here do not satisfy a reality condition

$$\bar{\Phi} \star \bar{\Psi} \neq \overline{\Psi \star \Phi}\,,$$

but they are perfectly consistent (for reality conditions on star products, see for example ref. [26]). As for the Poisson brackets we have

$$\bar{\Phi} \star_1 \bar{\Psi} = \overline{\Psi \star_2 \Phi}\,, \qquad \bar{\Phi} \star_3 \bar{\Psi} = \overline{\Psi \star_4 \Phi}\,.$$

4. Deformed extended superspace: chiral and harmonic superfields

We consider now the case of a superspace with extended supersymmetry, in which more star products exist. In particular, there exists a star product that is not only invariant under supertranslations but also Lorentz invariant. For definiteness we consider a Poisson bracket of type $\{\ ,\ \}_1$ with $P^{\mu\nu} = P^{\mu\alpha} = 0$. The same arguments could be used for brackets of type $\{\ ,\ \}_{2,3,4}$.

Let $i, j = 1, \ldots N$. We have

$$\{\Phi, \Psi\} = P^{i\alpha\, j\beta} \Phi \overleftarrow{D}_{\alpha i} \overrightarrow{D}_{\beta j} \Psi\,. \tag{4.1}$$

The matrix $P^{i\alpha\, j\beta}$ must be symmetric under the simultaneous exchange $i \leftrightarrow j$ and $\alpha \leftrightarrow \beta$. We can write

$$P^{i\alpha\, j\beta} = P_s^{ij\,\alpha\beta} + P_a^{ij} \epsilon^{\alpha\beta}\,, \tag{4.2}$$

where the subscripts a and s mean that the matrix is antisymmetric or symmetric respectively. The first term is symmetric under the independent exchanges $i \leftrightarrow j$ and $\alpha \leftrightarrow \beta$.

The second term is Lorentz invariant. Any Poisson bracket containing only the second term will be Lorentz invariant, and then super Poincaré invariant. This term has no analogue in $N = 1$.

We could choose $P_s^{ij\,\alpha\beta} = \delta^{ij} P_s^{\alpha\beta}$. Then first term would be O(N) invariant.

For $N = 2$ we could choose

$$P^{\alpha i\, \beta j} = P \epsilon^{ij} \epsilon^{\alpha\beta}\,.$$

Then, since ϵ^{ij} is an invariant form of SO(2) we can have SO(2) invariance and Lorentz invariance simultaneously. More generally, we can decompose (4.2) in terms of representations of SO(2)

$$P^{i\alpha\, j\beta} = \delta^{ij} P_{0s}^{\alpha\beta} + P_{Ts}^{ij\,\alpha\beta} + P_{0a} \epsilon^{ij} \epsilon^{\alpha\beta}\,,$$

where the first and last terms are SO(2) singlets and the second term is the SO(2) doublet corresponding to the traceless part, ($P_{Ts}^{ij\,\alpha\beta} \delta_{ij} = 0$). The second term does not exist in $N = 1$.

It is convenient to make the change of variables

$$\theta^{\pm\alpha} = \frac{1}{\sqrt{2}}\left(\theta^{1\alpha} \pm i\theta^{2\alpha}\right), \qquad \bar{\theta}^{\pm\dot{\alpha}} = \frac{1}{\sqrt{2}}\left(\theta^{1\dot{\alpha}} \pm i\theta^{2\dot{\alpha}}\right)$$

so that $(\theta^{+\alpha})^* = \bar{\theta}^{-\dot{\alpha}}$. The labels $\pm$ are charges under SO(2). (4.2) becomes

$$P^{\pm\alpha\,\mp\beta} = i(P_{0s}^{\alpha\beta} \mp P_{0a}\epsilon^{\alpha\beta})\,, \qquad P^{\pm\alpha\,\pm\beta} = i(P_{Ts}^{1\alpha\,1\beta} \pm P_{Ts}^{1\alpha\,2\beta})\,.$$

This is a suitable basis to describe harmonic superspace [22]. We make a shift in the superspace variables

$$x^\mu,\, \theta^\pm,\, \bar{\theta}^\pm \longrightarrow y^\mu = x^\mu + i\theta^+\sigma^\mu\bar{\theta}^- - i\theta^-\sigma^\mu\bar{\theta}^+,\, \theta^\pm,\, \bar{\theta}^\pm\,.$$

In this basis the covariant derivatives become

$$\begin{aligned} D_\alpha^+ &= \frac{\partial}{\partial\theta^{-\alpha}}\,, \qquad & D_\alpha^- &= -\frac{\partial}{\partial\theta^{+\alpha}} + 2i\bar{\theta}^{-\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \\ \bar{D}_{\dot{\alpha}}^+ &= \frac{\partial}{\partial\bar{\theta}^{-\dot{\alpha}}}\,, \qquad & \bar{D}_{\dot{\alpha}}^- &= -\frac{\partial}{\partial\bar{\theta}^{+\dot{\alpha}}} - 2i\theta^{-\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\,. \end{aligned}$$

The change of basis can be expressed as

$$D_\alpha^\pm = U_i^\pm D_\alpha^i\,,$$

with $U_i^\pm$ being harmonics in S^2. The only non zero anticommutators are

$$\{D_\alpha^+, \bar{D}_{\dot{\alpha}}^-\} = -\{D_\alpha^-, \bar{D}_{\dot{\alpha}}^+\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\,.$$

Chiral superfields satisfy

$$\bar{D}_{\dot{\alpha}}^\pm\Phi_c = 0\,,$$

while harmonic superfields are defined as

$$D_\alpha^-\Phi_h = \bar{D}_{\dot{\alpha}}^-\Phi_h = 0\,.$$

In $N = 2$ superspace, chiral superfields describe vector multiplets while harmonic superfields describe hypermultiplets.

One could choose for example a deformation like,

$$\begin{aligned} \{\Phi, \Psi\} = {} & P^{-\alpha,-\beta}\Phi\overleftarrow{D}_\alpha^+\overrightarrow{D}_\beta^+\Psi + P^{+\alpha,+\beta}\Phi\overleftarrow{D}_\alpha^-\overrightarrow{D}_\beta^-\Psi + \\ & + P^{+\alpha,-\beta}\Phi\overleftarrow{D}_\alpha^-\overrightarrow{D}_\beta^+\Psi + P^{-\alpha,+\beta}\Phi\overleftarrow{D}_\alpha^+\overrightarrow{D}_\beta^-\Psi\,. \end{aligned} \tag{4.3}$$

This has no analogue in $N = 1$, if $P^{-\alpha,-\beta}$ or $P^{+\alpha,+\beta}$ is different from zero. Nevertheless, we have as before that antichiral superfields have Poisson bracket zero, so their product is not deformed.

Another choice could be a deformation like

$$\begin{aligned} \{\Phi, \Psi\} = {} & P^{-\alpha,-\dot{\alpha}}\left(\Phi\overleftarrow{D}_\alpha^+\overrightarrow{\bar{D}}^+\dot{\alpha}\Psi + \Phi\overleftarrow{\bar{D}}^+\dot{\alpha}\overrightarrow{D}_\alpha^+\Psi\right) + \\ & + P^{-\alpha,-\alpha}\Phi\overleftarrow{D}_\alpha^+\overrightarrow{D}^+\alpha\Psi + P^{-\dot{\alpha},-\dot{\alpha}}\Phi\overleftarrow{\bar{D}}^+\dot{\alpha}\overrightarrow{\bar{D}}_{\dot{\alpha}}^+\Psi\,. \end{aligned} \tag{4.4}$$

Again, it has no analogue in $N = 1$. In this case the product in the subspace of *antiharmonic* superfields ($D_\alpha^+\Phi_{ah} = \bar{D}_{\dot{\alpha}}^+\Phi_{ah} = 0$) is not deformed.

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If only one term, say $P^{-\alpha,-\dot{\alpha}}$, is different from zero, then the deformation is given in terms of the vector $v^\mu = P^{-\alpha,-\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}$.

As in section 3.2, we could do the same analysis with the operators Q's instead of the operators D's.

In section 5.2 we will comment about the physical consequences of these deformations.

5. Wess-Zumino model in non commutative superspace

In this section we want to explore the possibility of using the star product to construct lagrangians of physical theories. The "purely even" deformation (that is, a deformation that is non trivial on the coordinates x^μ) has been studied in [14]. In order to see the effects of the non commutativity of the odd variables, we will set $P^{\mu\nu} = 0$. We will analyze the theory in both, euclidean and minkowskian signatures.

In the first place, we will consider a Poisson bracket of type $\{\ ,\ \}_1$ and its quantization, which is supersymmetric. A bracket of type $\{\ ,\ \}_3$ was used in ref. [20], with which we will compare our results.

From now on, unless explicitly stated the star product $\Phi\star\Psi$, without any subindex, will refer to the Moyal-Weyl quantization of $\{\ ,\ \}_1$ with $P^{\mu\nu} = P^{\mu\alpha} = 0$, so

$$\{\Phi,\Psi\} = P^{\alpha\beta}\Phi\overleftarrow{D}_\alpha\overrightarrow{D}_\beta\Psi\,.$$

The star product has a finite expansion

$$\begin{aligned}\Phi\star\Psi &= \Phi\Psi + hP^{\alpha\beta}\Phi\overleftarrow{D}_\alpha\overrightarrow{D}_\beta\Psi + \frac{h^2}{4}\det P\,\Phi\overleftarrow{D}^2\overrightarrow{D}^2\Psi\\ &= \Phi\Psi + hP^{\alpha\beta}(-1)^{(p_\Phi+1)}\overrightarrow{D}_\alpha\Phi\overrightarrow{D}_\beta\Psi - \frac{h^2}{4}\det P\,\overrightarrow{D}^2\Phi\overrightarrow{D}^2\Psi\,,\end{aligned} \tag{5.1}$$

where

$$\overleftarrow{D}^2 = \overleftarrow{D}_\alpha\overleftarrow{D}_\beta\epsilon^{\alpha\beta}\,,\qquad \overrightarrow{D}^2 = \epsilon^{\beta\alpha}\overrightarrow{D}_\alpha\overrightarrow{D}_\beta\,.$$

Another convenient way of expressing the star product is

$$\Phi\star\Psi = \Phi\Psi + \overrightarrow{D}_\alpha\left(hP^{\alpha\beta}(-1)^{(p_\Phi+1)}\Phi\overrightarrow{D}_\beta\Psi + \frac{h^2}{4}\det P\,\overrightarrow{D}^\alpha\Phi\overrightarrow{D}^2\Psi\right), \tag{5.2}$$

which makes manifest the fact that the difference between the ordinary product and the star product is a total covariant derivative.

From now on we will use only left derivatives and we will denote them simply as $D_\alpha, \bar{D}_{\dot{\alpha}}$. We will also consider only even superfields, so $p_\Phi = 0$,

The subalgebra of antichiral fields ($D\Phi_a = 0$) does not get deformed,

$$\Phi_a\star\Psi_a = \Phi_a\Psi_a\,.$$

More generally, the star product of an antichiral field with a general field is the commutative product,

$$\Phi_a\star\Psi = \Phi_a\Psi\,.$$

5.1 1/2 supersymmetric lagrangian in euclidean superspace

In this subsection we consider a superspace with euclidean signature, so θ and $\bar{\theta}$ are pseudoreal and independent. We will see that the difficulty in defining a star product that is simultaneously supersymmetric and has subalgebras of chiral and antichiral superfields will lead to an explicit breaking of supersymmetry in the lagrangian.

Let Φ be a chiral superfield, and $\bar{\Phi}$ an antichiral superfield. The prescription for the kinetic term in the Wess-Zumino lagrangian is

$$\int d^2\theta d^2\bar{\theta}\, \Phi \star \bar{\Phi} = D^2 \bar{D}^2 \Phi \star \bar{\Phi}|_{\theta=\bar{\theta}=0}$$

which does not get deformed. The interaction terms that involve powers $\bar{\Phi}^{(\star n)} = \bar{\Phi}^n$ (in an obvious notation) do not get deformed neither. These superfields are antichiral, so the classical prescription for the lagrangian

$$\int d^2\bar{\theta}\, \bar{\Phi}^n = \bar{D}^2 \Phi \star \bar{\Phi}|_{\bar{\theta}=0}$$

does not break supersymmetry.

We want to analyze the terms that can have a non trivial contribution from the star product, as for example, $\Phi^{(\star n)}$. This term is not a chiral superfield. But we can still give the prescription

$$\int d^2\theta\, \Phi^{\star n} = D^2 \Phi^{(\star n)}|_{\theta=\bar{\theta}=0}\,.$$

We will see that this prescription breaks 1/2 of the supersymmetries, the $\bar{Q}_\alpha$, but it is still invariant under the Q_α.

For $n = 2$, and using (5.2), we have

$$D^2(\Phi \star \Phi) = D^2(\Phi^2)\,.$$

For $n = 3$, and using (5.1) we have

$$D^2(\Phi \star \Phi \star \Phi) = D^2(\Phi^3) - \frac{h^2}{4} \det P D^2\Phi D^2\Phi D^2\Phi\,. \tag{5.3}$$

Let us express the chiral field in terms of ordinary fields as usual

$$\Phi(y, \theta) = A(y) + \theta\psi(y) + \theta\theta F(y)\,.$$

Then

$$D^2(\Phi \star \Phi \star \Phi)|_{\theta=\bar{\theta}=0} = D^2(\Phi^3)|_{\theta=\bar{\theta}=0} - \frac{h^2}{4} \det P F^3\,.$$

The term that is added to the action is proportional to F^3. Since the term $\Phi^{(\star 3)}$ is non chiral, the lagrangian cannot be invariant under the whole supersymmetry algebra. Let us see this statement in more detail. Let Ψ be an antichiral superfield $D\Psi = 0$,

$$\Psi(\bar{y}, \bar{\theta}) = B(\bar{y}) + \bar{\theta}\chi(\bar{y}) + \bar{\theta}\bar{\theta}G(\bar{y})\,.$$

The supersymmetry transformations are

$$\begin{aligned} \delta_\epsilon B &= 0 & \delta_{\bar\epsilon} B &= \bar\epsilon\bar\chi \\ \delta_\epsilon \bar\chi &= -i2\epsilon\sigma^\mu\partial_\mu B & \delta_{\bar\epsilon}\bar\chi &= 2G\bar\epsilon \\ \delta_\epsilon G &= i\epsilon\sigma^\mu\partial_\mu\bar\chi & \delta_{\bar\epsilon} G &= 0\,. \end{aligned} \tag{5.4}$$

The component B of an antichiral superfield is invariant under the supersymmetries Q. If $\bar\chi$ were a total derivative, then the integral of B on space time would be invariant under both, the Q and $\bar Q$ supersymmetries. In particular $D^2\Theta$ is an antichiral superfield for arbitrary Θ, and if Θ is itself a chiral superfield then the component $\bar\chi$ is a total derivative.

In this case, the lagrangian is constructed as $D^2\Theta$, but Θ contains non chiral terms. As a consequence, the action can only be invariant under half of the supersymmetry generators.

It is remarkable that the lagrangian that we obtain is the same that the one proposed in ref. [20]. There, the star product chosen was the star product quantizing a bracket of type $\{\ ,\ \}_3$ in (3.4). This star product breaks the $\bar Q$ supersymmetries explicitly, and although chiral fields are a good subalgebra of the star product, the resulting lagrangian preserves only half of the supersymmetries.

It is easy to compute some higher order terms in the lagrangian,

$$\begin{aligned} D^2(\Phi^{(\star 4)}) &= D^2\Phi^4 - \frac{1}{2}\det P(D^2\Phi^2)D^2\Phi D^2\Phi\,, \\ D^2(\Phi^{(\star 5)}) &= D^2\Phi^4 + \frac{1}{2}\det P(D^2\Phi)^2D^2\Phi^3 + \frac{1}{4}\det P(D^2\Phi^2)^2D^2\Phi + \frac{1}{16}(\det P)^2(D^2\Phi)^5\,. \end{aligned}$$

As mentioned in ref. [20], the terms appearing in the lagrangian contain $P^{\alpha\beta}$ only through the expression $\det P$, so the action is Lorentz invariant.

Quantum properties of the model with the deformation involving the Q_α operators have been studied in refs. [27, 28]

5.2 A comment on $N = 2$ theories

We remind that for $N = 2$ theories we considered two types of deformations, (4.3) and (4.4). One could construct theories with the method we have used for $N = 1$. In the first case (the chiral case), one would expect that the $\bar Q^\pm_{\dot\alpha}$ supersymmetry will be broken, while in the second case (the harmonic case) the broken supersymmetries will be $\bar Q^+_{\dot\alpha}, Q^+_\alpha, $.

Since $N = 2$ theories may contain both, chiral and harmonic superfields, such a theory will have three broken supersymmetries, $\bar Q^\pm_{\dot\alpha}$ and $\bar Q^+_{\dot\alpha}$. This is in agreement with ref. [21].

5.3 An example with $P^{\mu\alpha} \neq 0$

In this section we want to consider an example of a deformation with $P^{\mu\alpha} \neq 0$. Generically $P^{\mu\alpha} \neq 0$ contains 4×2 odd parameters, so the expansion of the star product (3.6) will necessarily end at order 8. We consider here the following ansatz: We take $P^{\mu\alpha} = 0$ for

$\mu \neq 1$ and $P^{1\alpha} \neq 0$. We denote $\partial_1 = \partial$ and $P^2 = P^{1\alpha}P_{1\alpha}$. We have

$$\begin{aligned}\Phi \star \Phi &= \Phi^2 + \frac{h^2}{2}P^2(D^\alpha\partial\Phi D_\alpha\partial\Phi - D^2\Phi\partial^2\Phi)\\ &= \Phi^2 + \frac{h^2}{2}P^2\big((D^\alpha(\partial\Phi\partial D_\alpha\Phi) - \partial(\partial\Phi D^2\Phi)\big)\\ &= \Phi^2 + \frac{h^2}{2}P^2\left(\frac{1}{2}D^2(\partial\Phi)^2 - \partial(\partial\Phi D^2\Phi)\right).\end{aligned}$$

We want to compute the contribution of $\Phi \star \Phi \star \Phi$ to the lagrangian,

$$\begin{aligned}D^2\big((\Phi\star\Phi)\Phi\big) &= D^2\left(\Phi^3 + \frac{h^2}{2}P^2\left(\frac{1}{2}D^2(\partial\Phi)^2\Phi - \partial(\partial\Phi D^2\Phi)\Phi\right)\right)\\ &\backsimeq D^2\Phi^3 + \frac{3}{2}\frac{h^2}{2}D^2(\partial\Phi)^2D^2\Phi^2\,,\end{aligned}$$

modulo terms that are total spacetime derivatives. The deformation term is proportional to

$$P^2F\left(\frac{1}{4}\partial\psi\partial\psi - \partial A\partial F\right) \backsimeq P^2\left(\frac{1}{4}(\partial\psi^\alpha\partial\psi_\alpha)F + \frac{1}{2}F^2\Box A\right).$$

The term of order 1 in h will appear only if we take more than one superfield. We could have, for example $(\Phi_1 \star \Phi_2)\Phi$. Assume for simplicity that only one Grassmann parameter $P^{\mu\alpha}$ is different from zero, so only the first order contributes. We keep nevertheless the Lorentz covariant notation. Then, up to total spacetime derivatives

$$\begin{aligned}D^2\big((\Phi_1\star\Phi_2)\Phi\big) &\backsimeq P^{\mu\alpha}\big(\Phi_1D_\alpha\Phi_2\partial_\mu\Phi - \Phi_1\partial_\mu\Phi_2D_\alpha\Phi\big)\\ &\backsimeq P^{\mu\alpha}\big(((A_1\psi_{2\alpha}) - (A_2\psi_{1\alpha}))\partial_\mu F - (A_1F_2 - A_2F_1)\partial_\mu\psi_\alpha +\\ &\quad + F(\psi_{1\alpha}\partial_\mu A_2 - \psi_{2\alpha}\partial_\mu A_1 + A_1\partial_\mu\psi_{2\alpha} - A_2\partial_\mu\psi_{1\alpha}) -\\ &\quad - (F_1\partial_\mu A_2 - F_2\partial_\mu A_1 + A_1\partial_\mu F_2 - A_2\partial_\mu F_1)\psi_\alpha\big).\end{aligned}$$

5.4 Supersymmetric lagrangian in minkowskian superspace

We consider now a superspace with minkowskian signature. We remind that the star product does not have a reality condition,

$$\bar\Phi \star \bar\Psi \neq \overline{\Psi\star\Phi}\,. \tag{5.5}$$

Φ and $\bar\Phi$ are now related by complex conjugation. The kinetic term $\Phi \star \bar\Phi = \Phi\bar\Phi$ is real and not deformed. We consider terms of two types, with the following prescriptions:

1. $\Phi^{(\star n)}$ is a non-chiral superfield. It contributes to the action as

$$\int d^2\theta d^2\bar\theta\,\Phi^{(\star n)} = D^2\bar D^2\Phi^{(\star n)}|_{\theta=\bar\theta=0}\,.$$

This becomes zero in the classical limit, $h \to 0$, so it is a purely non commutative correction. It is not real, so we follow the standard prescription of adding the hermitean conjugate term to the action,

$$\int d^2\theta d^2\bar\theta\,\overline{\left(\Phi^{(\star n)}\right)} = D^2\bar D^2\overline{\left(\Phi^{(\star n)}\right)}|_{\theta=\bar\theta=0}\,.$$

Notice that because of (5.5), $\overline{\left(\Phi^{(\star n)}\right)} \neq \bar\Phi^{(\star n)}$.

2. $\bar\Phi^{(\star n)} = \bar\Phi^n$ is an antichiral superfield. We introduce it in the action as

$$\int d^2\bar\theta\, \bar\Phi^n = \bar D^2 \bar\Phi^n|_{\bar\theta=0}\,.$$

Again, we should add the hermitean conjugate term

$$\int d^2\theta\, \overline{(\bar\Phi^n)} = D^2\overline{(\bar\Phi^n)}|_{\theta=0} = D^2(\Phi^n)|_{\theta=0}\,.$$

The kinetic term plus terms of the type 1 and 2 (together with their hermitean conjugates) give a deformed, supersymmetric Wess-Zumino model, formulated in a non commutative superspace with a non trivial deformation of the odd variables, and that has the correct classical limit.

The deformation of $\Phi^{(\star 2)} = \Phi^2$ is a total (∂, D) derivative.

Notice that one can use either the star product corresponding to a Poisson bracket of type $\{\ ,\ \}_1$ (involving the D's), or a star product associated with the Poisson bracket $\{\ ,\ \}_2$ (involving the $\bar D$'s). The prescription to compute the lagrangian will lead to the same lagrangian. So both star products can be used interchangeably in the action, and have the same physical meaning.

Let us compute the term of type 1. For $n = 3$. We have (using (5.3))

$$\begin{aligned}\bar D^2 D^2(\Phi\star\Phi\star\Phi)|_{\theta=\bar\theta=0} &= \bar D^2\Big(D^2(\Phi^3) - \frac{h^2}{4}\det P D^2\Phi D^2\Phi D^2\Phi\Big)|_{\theta=\bar\theta=0}\\ &= \frac{h^2}{4}\det P\left(\frac{1}{2}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}\partial_\mu\psi^\alpha\partial_\nu\psi^\beta F + F^2\Box A\right).\end{aligned}$$

Notice that the correction to the Wess-Zumino action computed in section 5.1 corresponds to the first component of the chiral superfield $(D^2\Phi)^3$, and consequently breaks 1/2 supersymmetry ($\bar Q$) (see (5.4)). In minkowskian spacetime, adding the hermitean conjugate of this term will break the other half of the supersymmetry (Q), so the resulting action will not be invariant under any supersymmetry.

Instead, the correction that we compute here is the last component of the same superfield $(D^2\Phi)^3$. The variation of the last component of a chiral superfield under supersymmetry transformations is a total spacetime derivative, and the action is supersymmetric.

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